

## DEPARTMENT OF MASTER OF BUSINESS ADMINISTRATION

# Linear Programming

## 2.1. LINEAR PROGRAMMING PROBLEMS(LPP)

### 2.1.1. Meaning and Definition

Linear programming is a two-word phrase. It means:

- 1) The word **linear** refers to any **linear** relationship among variables in a model. This means that any change in one variable will result into a proportional change in other variables.
- 2) **Programming** refers to any problem that can be modelled and solved mathematically. There are various alternate strategies to achieve the desired objective. But in programming, economic allocation of limited resources and choosing a particular course of strategy helps in solving the problem.

An analysis of problems represented in linear function with a number of variables that is to be optimised, when subjected to a number of restraints in the form of inequalities, is called **Linear Programming**.

A problem can be approached and solved with different strategies. Linear Programming selects the best possible strategy from the available alternatives. Strategy is selected on the basis of maximisation or minimisation of some required output, i.e., it has to be optimal.

**For example, maximisation of output/profit or minimisation of production cost.**

**According to William M Fox, “Linear programming is a planning technique that permits some objective functions to be minimised or maximised within the framework of given situational restrictions.”**

Linear programming is a technique that works on a mathematical basis for determining the optimal solution. It takes into consideration the alternative uses of resources like man, machine, money, material, etc., to attain a particular objective.

Linear programming can be applied on various problems. However, this technique can be ideally used for solving maximisation and/or minimisation problems, subject to some assumptions. **For example**, maximisation of profit/sales or minimisation of cost.

### 2.1.2. General Mathematical Model of LPP

Let  $x_1, x_2, \dots, x_n$  be  $n$  decision variables. Then the mathematical form of linear programming problem is as follows:

### Optimise (Maximise or Minimise)

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n \quad (\text{Objective function})$$

**Subject to the constraints:**

[illegible]

and  $x_1, x_2, \dots, x_n \geq 0$  (Non-negativity condition)

Where  $a_{ij}$ ,  $b_j$ ,  $c_j$  are constants and  $x_j$  is decision variable.

By using the symbol ' $\Sigma$ ', i.e., the 'sum' of notation, the above formulation may be put in the following compact form:

Optimise (Max. or Min.)

$$Z = \sum_{i=1}^m \sum_{j=1}^n c_j x_j \quad (\text{Objective function})$$

Subject to the linear constraints:

$$\sum_{j=1}^n a_{ij}x_j (\leq, =, \geq) b_i; i=1, 2, \dots, m \quad (\text{Constraints})$$

and  $x_j \geq 0; j = 1, 2, \dots, n$  (Non-negativity condition)

## Some Useful Definitions

**Definition 1:** A feasible Solution to the linear programming problem is a vector  $X = (x_1, x_2, x_3, \dots, x_n)$  which satisfies conditions (2) and (3).

A feasible solution to a linear program is a solution that satisfies all constraints.

**Definition 2:** A basic solution to equation (2) is a solution obtained by setting  $(n - m)$  variables equal to zero and solving for the remaining  $m$  variables, provided that the determinant of the coefficients of these  $m$  variables is non-zero. The  $m$  non-zero variables are called **basic variables** and  $(n - m)$  zero variables are called **non-basic variables**.



For examples,

1) **Maximisation Case**

$$\begin{aligned} \text{Maximise } Z &= 40x_1 + 35x_2 && \text{(Profit)} \\ \text{Subject to } 2x_1 + 3x_2 &\leq 60 && \text{(First constraint)} \\ &4x_1 + 3x_2 \leq 96 && \text{(Second constraint)} \\ &x_1, x_2 \geq 0 && \text{(Non-negativity condition)} \end{aligned}$$

2) **Minimisation Case**

$$\begin{aligned} \text{Minimise } Z &= 40x_1 + 24x_2 && \text{(Cost)} \\ \text{Subject to } 20x_1 + 50x_2 &\geq 4800 && \text{(First constraint)} \\ &80x_1 + 50x_2 \geq 7200 && \text{(Second constraint)} \\ &x_1, x_2 \geq 0 && \text{(Non-negativity condition)} \end{aligned}$$

### Basic Feasible Solution

A basic feasible solution to a LPP is called as a basic feasible solution if it satisfies the non-negative restriction. There are two types of basic feasible solutions:

- 1) **Non-degenerate:** All  $m$  basic variables are positive and remaining  $n$  variables will be zero.
- 2) **Degenerate:** A basic feasible solution is degenerate, if one or more basic variables are zero.

## 2.1.3. Assumptions of Linear Programming Model

Four basic assumptions that are important for all linear programming problems are as follows:

- 1) **Certainty:** All the variables or parameters in an LP model should be assumed to be constant or known. For the best optimal solution, parameters such as availability and consumption of resources or the profit/cost contribution variable should be assumed to be certain. Certainty in an LP model is concerned with the coefficients in the objective function  $c_j$ , the coefficients in the functional coefficients  $a_{ij}$  and the right hand sides of the functional constraints  $b_i$ . The values of the coefficients of each variable are to be constant or certain. This means that values of  $c_j$ ,  $a_{ij}$ ,  $b_i$  are fixed and known with certainty.
- 2) **Divisibility (or Continuity):** It is assumed that decision variables and resources have solution values which are either whole numbers (integers) or mixed numbers (integer and fractional). The integer programming method may be applied to get the desired values if only integer variables are desired, e.g., machines, employees, etc. However, in an LPP model, decision variables can have any value that satisfy the functional and non-negative constraints. It can also take a non-integer value.
- 3) **Additive:** This means that the function value is the sum of the contributions of each term, i.e., when two or more activities are used, the total product is equal to the sum of the individual products where there is no interaction effect between the activities.

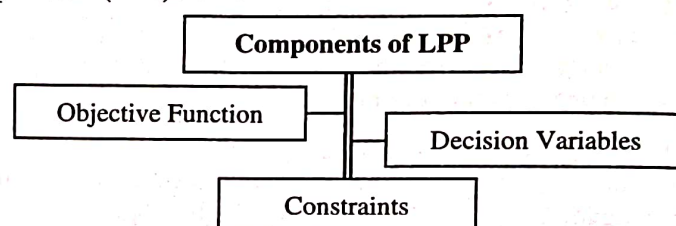
For example, the sum of the profits earned separately from A and B must be equal to the total profit earned by the sale of two products A and B. Similarly, the sum of resources used for A and B individually must be equal to the amount of a resource consumed by A and B.

- 4) **Linearity:** Linear programming requires linearity in the equations, i.e., the relationship in both objective function and constraints must be linear. Linearity requires that a change in a variable should result in proportionate change in that variable's contribution to the value of the function.

For example, the resource  $i$  consumed is  $5a_{ij}$ , if decision variable  $x = 5$  but the consumption will be  $10a_{ij}$  if  $x = 10$ , and so on. Where  $a_{ij}$  represents the amount of resource  $i$  used for an activity  $j$  (decision variable).

## 2.1.4. Components of Linear Programming Problem

Following are the components of a linear programming problem (LPP):



- 1) **Objective Function:** LPP has a component called Objective Function or Criterion function. It is a linear function and can work either by maximisation or minimisation process. It includes all possible components of a problem to optimise the solution.
- 2) **Decision Variables:** The objective function uses variables to arrive at a solution. The variables that help to decide the outcome are called 'Decision Variables' or 'Activity Variables'. The level of activity for each variable is specified. However, it can specify a zero value of some variable but not a negative value. Decision (or choice) variables can be  $x_1, x_2, \dots, x_n$ .
- 3) **Constraints:** Constraints are a part of every practical situation. The mathematical program stated in an algebraic form will also specify constraints, i.e., the limited availability of resources, if it is a maximisation problem or a minimum quality or composition, if it is a minimisation problem. Thus, every problem will come with its limitations and accordingly, a strategy or an approach to a problem is to be developed. This is called a Constraint.

Constraint is expressed as ( $<$ ,  $=$ ,  $>$ ) follows:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 \\ a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\geq b_1 \text{ Or} \\ -a_{11}x_1 - a_{12}x_2 - \dots - a_{1n}x_n &\leq b_1 \end{aligned}$$

## 2.1.5. Generalized LPP

When a production system is being modelled, it may happen that the input and output coefficients of one or more activities are not in fixed proportions (as is the case for linear programs), but each column of coefficient may be freely chosen as any point from a convex set  $C_j$ .



This important class of problems is called a "Generalised Linear Program". These were first studied by Philip Wolfe.

**Definition:** A generalised linear program is a problem stated in standard linear programming problem format:

$$\begin{array}{ll} \text{Minimise} & C^T x = z \\ \text{Subject to} & Ax = b, \quad A: m \times n, \quad \dots (1) \end{array}$$

where each column  $\begin{pmatrix} c_j \\ A_{\bullet j} \end{pmatrix}$  may be freely chosen to be any point in a given convex set  $C_j$ ,  $j = 1, \dots, n$ .

By simple extension, the fixed right-hand-side vector  $b$  may also be replaced by a vector picked from a convex set  $C_b$ .

### Theorem 10.1: (Equivalent Generalised Linear Program)

The generalised linear program equation (2) is equivalent to the generalised equation (3) generated at some iteration of Wolfe's algorithm.

#### Original Generalised Linear Program:

$$\begin{array}{ll} \text{Minimise} & \sum_{j=1}^n c_j \hat{x}_j = \hat{z} \\ \text{Subject to:} & \sum_{j=1}^n A_{ij} \hat{x}_j = b_i, \text{ for } i = 1, \dots, m \dots (2) \\ & \hat{x}_j \geq 0, \text{ for } j = 1, \dots, n, \end{array}$$

where

$$\begin{pmatrix} c_j \\ A_{\bullet j} \end{pmatrix} \in C_j \text{ are freely chosen vectors in convex sets } C_j.$$

#### Equivalent Generalised Linear Program

$$\begin{array}{ll} \text{Minimise} & \sum_{j=1}^n c_j x_j + \sum_{t=1}^{T_j} c_j^t x_j^t = z \\ \text{Subject to:} & \sum_{j=1}^n \left( A_{ij} x_j + \sum_{t=1}^{T_j} A_{ij}^t x_j^t \right) = b_i, \text{ for } i = 1, \dots, m \dots (3) \\ & x_j \geq 0, \text{ for } j = 1, \dots, n, \end{array}$$

where  $\begin{pmatrix} c_j \\ A_{\bullet j} \end{pmatrix} \in C_j$  are freely chosen vectors in convex sets

$$C_j \text{ and } \begin{pmatrix} c_j^t \\ A_{\bullet j}^t \end{pmatrix} \in C_j, \quad t = 1, \dots, T_j \text{ and } T_j \text{ fixed points in } C_j.$$

## 2.1.6. Advantages of Linear Programming Model

- 1) **Improves Quality Decision:** LP improves the quality of the decisions. There might be other constraints operating outside the problem. Hence, linear programming gives possible and practical solution.

- 2) **Cost-Benefit Analysis:** LP is a versatile technique that helps in representing real-life business situations. It is a very cost-effective technique and is helpful in planning and executing the policies of the top management in a cost effective manner.
- 3) **Flexibility:** LP provides better tools for meeting the changing conditions. Re-evaluation for changing conditions can be done, even after the plans are prepared. LPP is a very effective technique under such changing circumstances.
- 4) **Number of Possible Solutions:** Management problems are complex, but with LPP technique, managers can arrive at the best alternative solution to the problem. LPP helps in assuring that the manager is considering the best optimal solution.
- 5) **Use of Productive Factors:** The Linear Programming technique helps in making the best possible use of the available productive resources such as time, labour, machine etc. A decision-maker can employ his productive factors effectively by selecting and distributing these elements.
- 6) **Scientific Approach:** LP is effective as it highlights the bottlenecks in the production process. This means that an LP presents a clear picture of the problem. Hence, it becomes easy to deal with the problem.

## 2.1.7. Limitations of Linear Programming Model

- 1) **Linear Relationship:** A primary requirement of an LP is that the objective function and every constraint must be linear. However, in real life situations, many business problems can only be expressed in a non-linear form. In such situations, LP technique is not applicable.
- 2) **Coefficients are Constraints:** LP assumes that all values of coefficients of decision variables are stated with certainty. Due to this restriction, LP cannot be applied to a wide variety of problems.
- 3) **Fractional Solutions:** Many times the solution to a problem may not be an integer but a fraction. Solution in fractions may not remain optional in rounding off.
- 4) **Complexity:** There are computational difficulties when it comes to large problems. LP model is a mathematical formulation which becomes complex when there are large number of variables and constraints.
- 5) **Possibility of More than One Objective:** LP deals with the problems with single objective. But in real life situations there are more than one objective. LP faces limitations in such situations.
- 6) **Time Effect:** The effect of time is not considered in linear programming model.

## 2.1.8. Application Areas of Linear Programming

Linear programming models can be applied to a variety of business problems. Following areas make use of this technique:



- 1) **Finance:** LP techniques are very important for the finance sector. It is very useful in profit planning and control. LP can be used to maximise the profit margins by making an optimum use of financial resources.
- 2) **Industrial:** Production costs can be put under check with the help of linear programming. LP finds the best solution under prevailing constraints and helps maximising output and minimising costs.
- 3) **Administrative:** The administrative tasks become easier and efficient with the use of LP techniques. It helps the manager to choose and decide the best method out of various managerial methods for achieving the desired output.
- 4) **Defence:** LP is extensively used in military operations. In the defence sector, LP can be used to utilise the optimal limited defence resources to achieve the desired goal.
- 5) **Trade:** Linear programming can be used very widely in this sector. It helps in the estimation of demand and supply, price and cost etc.
- 6) **Transport:** In the transport sector, LP models are employed to determine the optimal distribution system. It also helps in ascertaining that minimum cost is incurred on transportation.

### 2.1.9. Guidelines for Formulation of Linear Programming Model

To denote a general linear programming problem in mathematical form using various symbols and mathematical model is known as **Formulation of Linear Programming Problem**.

#### Steps of LPP Formulation

LP requires the formulation of a model in a mathematical form using various symbols. The basic steps in formulating a linear programming model are as follows:

**Step 1:** Identifying the decision variables (values of which are found by solving the L.P.P.) and assigning the symbols  $x_1, x_2, \dots$  or  $x, y, \dots$  to them.

**Step 2:** Identify the objective function which is optimised (minimised or maximised) and expressed in terms of pre-defined decision variables as below:

$$Z = c_1x_1 + c_2x_2 + \dots c_nx_n$$

**Step 3:** Identify all the constraints in a given problem, which restrict the operation at a given point of time and express them as linear equations and/or inequalities in terms of pre-defined decision variables as below:

$$a_{i1}x_1 + a_{i2}x_2 + \dots (\leq, =, \geq) b_i, \text{ where } i = 1, 2, \dots, m$$

**Step 4:** Since the negative values of decision variables do not have any valid physical interpretation, write the non-negativity conditions as:

$$x_1 \geq 0, x_2 \geq 0, \dots \text{ or } x_j \geq 0, \text{ where } j = 1, 2, \dots, n$$

**Step 5:** In the last stage, form a linear programming problem model by putting the objective function, linear constraints and non-negativity conditions together in the form of an equation.

### 2.1.10. Formulation of Linear Programming Problem

**Example 1:** A firm uses lathes, milling machines and grinding machines to produce two machine parts. Table given below represents the machining times required for each part, the machining times available on different machines and the profit on each machine part:

Type of Machine	Machining Time Required for the Machine Part (Minutes)		Maximum Time Available per Week (Minutes)
	I	II	
Lathes	12	6	3,000
Milling Machines	4	10	2,000
Grinding Machines	2	3	900
Profit per Unit	₹40	₹100	

Formulate the problem so that the number of parts I and II to be manufactured per week to maximise the profit.

#### Solution: Formulation of L.P. Model

- 1) **Objective Functions:** The first major requirement of an LPP that we should be able to identify the objective functions. It will be maximised or minimised. Mathematically the objective function relates the variables which we are dealing in the problem. In this problem we could be obtained by producing and selling two machine parts ( $M_1$ ) and ( $M_2$ ).

Let  $x_1$  and  $x_2$  represent the number of time required for producing and selling  $M_1$  and  $M_2$  respectively. Now, Machine part ( $M_1$ ) is obtained profit ₹40 and Machine part ( $M_2$ ) is obtained profit ₹100 respectively.

So, objective is to maximise the profit.

$\text{Max } (Z) = 40x_1 + 100x_2$  is the objective function.

- 2) **Constraints/Conditions (Constraints are on the Time Available on Each Machine):** The mathematical relationship which is used to explain the inequality in the variables. The inequalities can be expressed in terms of less than or equal ( $\leq$ ) and in the term of greater than or equal ( $\geq$ ).

Each part of machine ( $M_1$ ) required 12 minutes for Lathes and Machine ( $M_2$ ) require 6 minutes for Lathes. The total maximum time is available of Lathes machine is 3,000 minutes.

We can express the constraints for Lathes as:

$$12x_1 + 6x_2 \leq 3,000$$

Each of machine part ( $M_1$ ) require 4 minutes and machine part ( $M_2$ ) require 10 minutes for milling machines and total maximum time available is 2,000 minutes.

We can express the constraints for milling machines as:  $4x_1 + 10x_2 \leq 2,000$



Each of machine part ( $M_1$ ) will require 2 minutes and machine part ( $M_2$ ) will require 3 minutes for grinding machine and total maximum time available is 900 minutes.

We can express the constraints as follows:

$$2x_1 + 3x_2 \leq 900$$

Therefore, the subjective function is:

$$\text{For Lathes, } 12x_1 + 6x_2 \leq 3,000$$

$$\text{For milling machines } 4x_1 + 10x_2 \leq 2,000$$

$$\text{For grinding machine } 2x_1 + 3x_2 \leq 900$$

- 3) **Non-Negative Conditions:**  $x_1$  and  $x_2$  are the number of machine parts produced by machine ( $M_1$ ) and machine ( $M_2$ ) and they cannot have negative value. The non-negative condition is expressed as:

$$x_1, x_2 \geq 0$$

Now, we can write the problem in complete form of LPP as follows:

$$\text{Max (Z)} = 40x_1 + 100x_2 \quad (\text{Objective functions})$$

Subject to,

$$12x_1 + 6x_2 \leq 3,000$$

$$4x_1 + 10x_2 \leq 2,000 \quad (\text{Subjective functions})$$

$$2x_1 + 3x_2 \leq 900$$

and non-negative conditions is  $x_1, x_2 \geq 0$ .

**Example 2:** A firm is engaged in producing two products  $P_1$  and  $P_2$ . Each unit of product  $P_1$  requires 2kg of raw material and 4 labour hours for processing. Whereas each unit of product  $P_2$  requires 5kg of raw material and 3 labour hours of the same type.

Every week the firm has the availability of 50kg of raw material and 60 labour hours. One unit of product  $P_1$  sold earn profit ₹20 and unit of product  $P_2$  sold gives ₹30 as a profit. Formulate this problem as linear programming to determine as to how many units of each of the products should be produced per week so that the firm can earn maximum profit, assume all units produced can be sold in the market?

**Solution:** Let the company produced two types of product  $P_1(x_1)$  and  $P_2(x_2)$ . Now the problem is formulated as follows:

	Products		
Resources	$P_1(x_1)$	$P_2(x_2)$	Availability
Raw Material	2kg	5kg	50kg
Labour	4 hours	3 hours	60 hours
Profit	₹20	₹30	

- 1) **Objective Functions:** The first major requirement of an LPP is that we should be able to identify the objective function. Mathematically the objective function relates the variable which we are dealing in the problem. In this problem we could be obtained by producing and selling product  $P_1$  and  $P_2$ .

Let,  $x_1$  and  $x_2$  represent the number of units of product  $P_1$  and  $P_2$  respectively.

Max (Z) =  $20x_1 + 30x_2$  is the objective function.

- 2) **Constraints/Conditions:** The mathematical relationship which is used to explain the inequality in the variables. The inequality can be expressed in terms of less than or equal ( $\leq$ ) and in terms of greater than or equal ( $\geq$ ). Each unit of product  $P_1$  require 2kg of raw material while each unit of product  $P_2$  requires 5kg. The total availability of raw material is 50kg.

Now, we can express the constraints or condition as:

$$2x_1 + 5x_2 \leq 50$$

Similarly that a unit of product  $P_1$  requires 4 labour hours for its production and one unit of product  $P_2$  requires 3 labour hours with an availability of 60 labour hours. The labour constraints or condition will be expressed as mathematical equation:

$$4x_1 + 3x_2 \leq 60$$

Therefore, the subjective function is as follows:

$$2x_1 + 5x_2 \leq 50$$

$$4x_1 + 3x_2 \leq 60$$

- 3) **Non-Negative Conditions:**  $x_1$  and  $x_2$  the number of units produced of product  $P_1$  and  $P_2$  cannot have negative value. The non-negative condition is expressed as follows:

$$x_1, x_2 \geq 0$$

Now, we can write the problem in complete form of LPP as follows:

$$\text{Max } Z = 20x_1 + 30x_2$$

Subject to,

$$2x_1 + 5x_2 \leq 50$$

$$4x_1 + 3x_2 \leq 60 \text{ and}$$

$$x_1, x_2 \geq 0$$

**Example 3:** Mohan-Meaking Breweries Ltd. has two bottling plants, one located at Solan and the other at Mohan Nagar. Each plant produces three drinks, whisky, beer and fruit juices named A, B and C respectively. The numbers of bottles produced per day are as follows:

	Plant at	
	Solan(S)	Mohan Nagar (M)
Whisky, A	1,500	1,500
Beer, B	3,000	1,000
Fruit Juices, C	2,000	5,000

A market survey indicates that during the month of April, there will be a demand of 20,000 bottles of whisky, 40,000 bottles of beer and 44,000 bottles of fruit juices. The operating costs per day for plants at Solan and Mohan Nagar are 600 and 400 monetary units. Formulate the problem so that each plant be run in April so as to minimize the production cost, while still meeting the market demand?

**Solution: Formulation of L.P. Model**

- 1) **Objective Function:** Key decision is to determine the number of days for which each plant must be run in April. Let the plants at Solan and Mohan Nagar be run for  $x_1$  and  $x_2$  days.

Objective is to minimize the production cost.  
i.e., minimise  $Z = 600x_1 + 400x_2$



- 2) **Constraints:** Constraints are on the demand, i.e., for whisky,  $1,500 x_1 + 1,500 x_2 \geq 20,000$ ,  
For beer,  $3,000 x_1 + 1,000 x_2 \geq 40,000$ ,  
For fruit juices,  $2,000 x_1 + 5,000 x_2 \geq 44,000$
- 3) **Non-Negative Conditions:** Feasible alternatives are sets of values of  $x_1 \geq 0$  and  $x_2 \geq 0$  which meet the objective.

**Example 4:** A chemical company produces two products, X and Y. Each unit of product X requires 3 hours on operation I and 4 hours on operation II, while each unit of product Y requires 4 hours on operation I and 5 hours on operation II. Total available time for operations I and II is 20 hours and 26 hours respectively.

The production of each unit of product Y also results in two units of a by-product Z at no extra cost. Product X sells at profit of ₹10/unit, while Y sells at profit of ₹20/unit. By product Z brings a unit profit of ₹6 if sold; in case it cannot be sold, the destruction cost is ₹4/unit. Forecasts indicate that not more than 5 units of Z can be sold. Formulate the problem so that the quantities of X and Y to be produced, keeping Z in mind, so that the profit earned is maximum.

**Solution: Formulation of L.P. Model**

**Step 1:** The key decision to be made is to determine the number of units of products X, Y and Z to be produced.

**Step 2:** Let the number of units of products X, Y and Z produced be  $x_1, x_2, x_3$ , where

$x_3$  = number of units of Z produced  
= number of units of Z sold + number of units of Z destroyed =  $x_3 + x_4$  (say).

**Step 3:** Feasible alternatives are sets of values of  $x_1, x_2, x_3$  and  $x_4$ , where  $x_1, x_2, x_3, x_4 \geq 0$ .

**Step 4:** Objective is to maximise the profit. Objective function (profit function) for products X and Y is linear because their profits (₹10/unit and ₹20/unit) are constants irrespective of the number of units produced. A graph between the total profit and quantity produced will be a straight line. However, a similar graph for product Z is non-linear since it has slope + 6 for first part, while a slope of -4 for the second.

However, it is piece-wise linear, since it is linear in the regions (0, 5) and (5, Y). Thus splitting  $x_z$  into two parts, viz. the number of units of Z sold ( $x_3$ ) and number of units of Z destroyed ( $x_4$ ) makes the objective function for product Z also linear.

Thus the objective function is maximise  $Z = 10x_1 + 20x_2 + 6x_3 - 4x_4$ .

**Step 5:** Constraints are:

On time available on operation I:  $3x_1 + 4x_2 \leq 20$ ,

On time available on operation II:  $4x_1 + 5x_2 \leq 26$ ,

On the number of units of product Z sold:  $x_3 \leq 5$ ,

On the number of units of product Z produced:

$2x_2 = x_3 + x_4$  or  $-2x_2 + x_3 + x_4 = 0$  and

$x_1, x_2, x_3, x_4, x_5 \geq 0$

**Example 5:** A person wants to decide the constituents of diet which will fulfill his daily requirements of proteins, fat and carbohydrates at the minimum cost.

The choice is to be made from four different types of food. The yields per unit of these foods are given below:

Food Type	Yield per Unit			Cost per Unit
	Proteins	Fats	Carbohydrates	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum Requirement	800	200	700	

Formulate the LPP for the problem.

**Solution: Formulation of Linear Programming Model**

**Step 1)** Let consider that  $x_1, x_2, x_3$  and  $x_4$  are the number of units of food of type 1, 2, 3 and 4 to be used.

**Step 2)** The sets of values  $x_j$  show the feasible alternatives. Here  $x_j$  represents the number of units of food type j to be used. The values of j are 1, 2, 3 and 4.

Also  $x_j \geq 0$

...(1)

This shows non-negativity condition.

**Step 3)** The main objective is to minimise the total cost of foods, thus we have the following objective function:

Min  $Z = ₹(45x_1 + 40x_2 + 85x_3 + 65x_4)$

...(2)

**Step 4)** Daily requirements are fulfilled by various constraints which are given follows:

$3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$  (Proteins)

$2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$  (Fats)

$6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$  (Carbohydrates) ... (3)

Thus the objective of the linear programming models is to find the number of units  $x_1, x_2, x_3$  and  $x_4$  for minimising the objective function (equation 2) subject to constraints (equation 3) and follow non-negativity condition (equation 1).

**Example 6:** An agriculturist has a 125 acre farm. He produces radish, muttar and potato. Whatever he raises is sold fully in the market. He gets ₹5 per kg for radish, ₹4 per kg for muttar and ₹5 per kg for potato. The average per acre yield is 1500kg of radish, 1800kg of muttar and 1200kg of potato. To produce each 100 kg of radish and muttar and 80kg of potato, a sum of ₹12.50 has to be used for manure. Labour required for each acre to raise the crop is 6 man-days for radish and potato each and 5 man-days for muttar. A total of 500 man-days of labour at a rate of ₹40 per man-day is available.

Formulate this as a linear programming model to maximise the agriculturist's total profit. (10)

**Solution:** Let  $x_1, x_2$  and  $x_3$  be the number of acres allotted for cultivating radish, muttar and potato respectively.



Since the average yield of radish is 1,500kg per acre, and the selling price for radish is ₹5/kg hence the selling amount which the agriculturist gets from one acre is:  
 $₹5 \times 1,500\text{kg}$  Or ₹7,500

To produce 100kg of radish, the manure cost is ₹12.50, so the manure cost per acre will be:  
 $₹12.50 \times 1,500\text{kg}$

100kg  
 Or, ₹187.50

Labour cost per acre for radish:  
 $₹40 \times 6$  man days Or, ₹240

Profit per acre for radish:  
 $₹7,500 - ₹187.50 - ₹240$  Or, ₹7,072.50

Similarly, the selling price, manure cost, labour cost and profit per acre of land for mutter and potato are also calculated and presented in the following table:

Per Acre	Radish	Mutter	Potato
Selling Price	₹7,500 (₹5 × 1,500kg)	₹7,200 (₹4 × 1,800kg)	₹6,000 (₹5 × 1,200kg)
Less: Manure Cost	₹187.50 ( $\frac{₹12.50 \times 1,500\text{kg}}{100\text{kg}}$ )	₹225 ( $\frac{₹12.50 \times 1,800\text{kg}}{100\text{kg}}$ )	₹187.50 ( $\frac{₹12.50 \times 1,200\text{kg}}{80\text{kg}}$ )
Less: Labour Cost	₹240 (₹40 × 6 man days)	₹200 (₹40 × 5 man days)	₹240 (₹40 × 6 man days)
Profit	₹7,072.50	₹6,775	₹5,572.50

Since, the agriculturist wants to maximise the total profit, hence the objective function of the problem is given by:

$$Z = 7,072.50x_1 + 6,775.00x_2 + 5,572.50x_3$$

The linear programming model for the problem:

$$\text{Maximise } Z = 7,072.50x_1 + 6,775.00x_2 + 5,572.50x_3$$

Subject to the constraints:

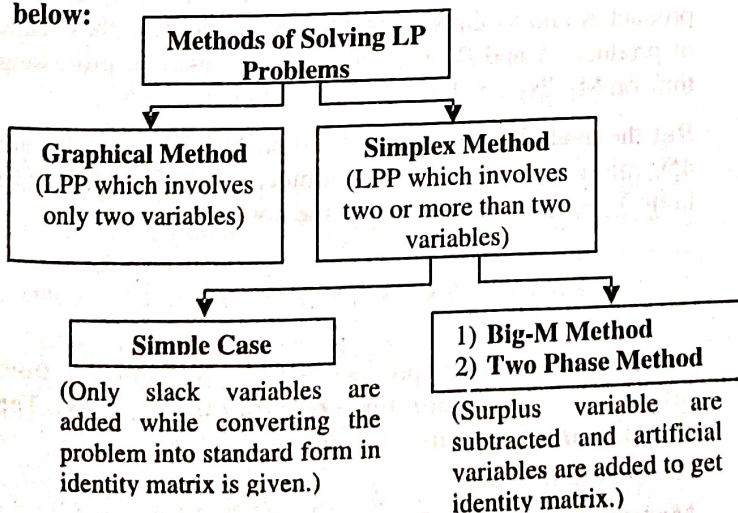
$$x_1 + x_2 + x_3 \leq 125$$

$$6x_1 + 5x_2 + 6x_3 \leq 500$$

Where,  $x_1, x_2, x_3 \geq 0$

### 2.1.11. Methods of Solving LP Problems

Linear Programming problem (LPP) can be basically solved by two methods. This is shown in the figure below:



Refine the solutions through repeated method. 25

- Graphical Method:** Graphical method is a very basic method of solving linear programming problems. Equations are represented in a graphical form and are solved using the intersection points and finding the values of the bounded area.
- Simplex Method:** The most general and powerful technique to solve linear programming problem is Simplex method. It is an iterative method which solves the LPP in a limited number of steps or indicates that the problem has unbounded solutions.

Simplex method is designed to solve a number of linear equations simultaneously with more/less unknown variables. The computational procedure is repeated until an optimal solution is determined.

## 2.2. OPTIMAL AND FEASIBLE SOLUTIONS BY GRAPHICAL METHOD

### 2.2.1. Introduction

Graphical method indicates the constraints and determines the 'feasible region' on the graph. The **feasible region** is the area that contains all possibly feasible solutions to the problem, i.e., those solutions which satisfy all the constraints of the problem.

The graphical method is applicable to solve the linear programming problem which involves two decision variables. One can arrive at an optimal solution to LPP by evaluating the value of the objective function at each vertex of the feasible region. It will only occur at one of the extreme points.

**For example,** let consider the following LP problem:  
 Maximize  $Z = 4x + 3y$

Subject to the restrictions,

$$x + y \leq 5$$

$$x + 2y \leq 8 \text{ and } x_1, x_2 \geq 0$$

In the figure 2.1, points B, C, D, E is vertices for the feasible region shaded area.

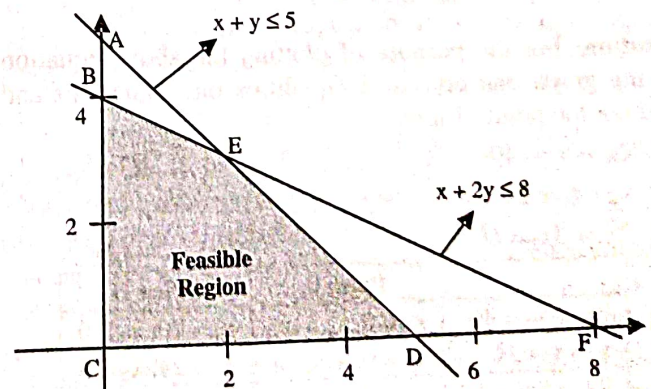


Figure 2.1